

Integer Concave Cocirculations and Honeycombs

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Abstract

A *convex triangular grid* is represented by a planar digraph G embedded in the plane so that (a) each bounded face is surrounded by three edges and forms an equilateral triangle, and (b) the union \mathcal{R} of bounded faces is a convex polygon. A real-valued function h on the edges of G is called a *concave cocirculation* if $h(e) = g(v) - g(u)$ for each edge $e = (u, v)$, where g is a concave function on \mathcal{R} which is affinely linear within each bounded face of G .

Knutson and Tao [4] proved an integrality theorem for so-called *honeycombs*, which is equivalent to the assertion that an integer-valued function on the boundary edges of G is extendable to an integer concave cocirculation if it is extendable to a concave cocirculation at all.

In this paper we show a sharper property: for any concave cocirculation h in G , there exists an integer concave cocirculation h' satisfying $h'(e) = h(e)$ for each boundary edge e with $h(e)$ integer and for each edge e contained in a bounded face where h takes integer values on all edges.

On the other hand, we explain that for a 3-side grid G of size n , the polytope of concave cocirculations with fixed integer values on two sides of G can have a vertex h whose entries are integers on the third side but $h(e)$ has denominator $\Omega(n)$ for some interior edge e . Also some algorithmic aspects and related results on honeycombs are discussed.

Keywords: Planar graph, Lattice, Discrete convex function, Honeycomb

AMS Subject Classification: 90C10, 90C27, 05C99

1 Introduction

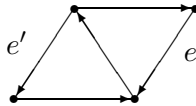
Knutson and Tao [4] proved a conjecture concerning highest weight representations of $\mathrm{GL}_n(\mathbb{C})$. They used one combinatorial model, so-called *honeycombs*, and an essential part of the whole proof was to show the existence of an integer honeycomb under prescribed integer boundary data. The obtained integrality result for honeycombs admits a re-formulation in terms of discrete concave functions on triangular grids in the plane.

The purpose of this paper is to show a sharper integrality property for discrete concave functions.

We start with basic definitions. Let ξ_1, ξ_2, ξ_3 be three affinely independent vectors in the euclidean plane \mathbb{R}^2 , whose sum is the zero vector. By a *convex (triangular) grid* we mean a finite planar digraph $G = (V(G), E(G))$ embedded in the plane such that: (a) each bounded face of G is a triangle surrounded by three edges and each edge (u, v) satisfies $v - u \in \{\xi_1, \xi_2, \xi_3\}$; and (b) the region $\mathcal{R} = \mathcal{R}(G)$ of the plane spanned by G is a convex polygon. In this paper a convex grid can be considered up to an affine transformation, and to visualize objects and constructions in what follows, we will fix the generating vectors ξ_1, ξ_2, ξ_3 as $(1, 0)$, $(-1, \sqrt{3})/2$, $(-1, -\sqrt{3})/2$, respectively. Then each bounded face is an equilateral triangle (a *little triangle* of G) surrounded by a directed circuit with three edges (a *3-circuit*). When \mathcal{R} forms a (big) triangle, we call G a *3-side grid* (this case is most popular in applications).

A real-valued function h on the edges of G is called a *cocirculation* if the equality $h(e) + h(e') + h(e'') = 0$ holds for each 3-circuit formed by edges e, e', e'' . This is equivalent to the existence of a function g on \mathcal{R} which is affinely linear within each bounded face and satisfies $h(e) = g(v) - g(u)$ for each edge $e = (u, v)$. Such a g is determined up to adding a constant, and we refer to h as a *concave cocirculation* if g is concave. (The restriction of such a g to $V(G)$ is usually called a *discrete concave function*.) It is easy to see that a cocirculation h is concave if and only if each *little rhombus* ρ (the union of two little triangles sharing a common edge) satisfies the following *rhombus condition*:

$$(1.1) \quad h(e) \geq h(e'), \text{ where } e, e' \text{ are non-adjacent (parallel) edges in } \rho, \text{ and } e \text{ enters an obtuse vertex of } \rho.$$



Let $B(G)$ denote the set of edges in the boundary of G . Concave cocirculations are closely related (via Fenchel's type transformations) to honeycombs, and Knutson and Tao's integrality result on the latter is equivalent to the following.

Theorem 1.1 [4] *For a convex grid G and a function $h_0 : B(G) \rightarrow \mathbb{Z}$, there exists an integer concave cocirculation in G coinciding with h_0 on $B(G)$ if h_0 is extendable to a concave cocirculation in G at all.*

For a direct proof of this theorem for 3-side grids, without appealing to honeycombs, see Buch [1]. (Note also that for a 3-side grid G , a combinatorial characterization for the set of functions h_0 on $B(G)$ extendable to concave cocirculations is given in [5]: it is a polyhedral cone in $\mathbb{R}^{B(G)}$ whose nontrivial facets are described by Horn's type inequalities with respect to so-called *puzzles*; for an alternative proof and an extension to arbitrary convex grids, see [3].)

In this paper we extend Theorem 1.1 as follows.

Theorem 1.2 *Let h be a concave cocirculation in a convex grid G . There exists an integer concave cocirculation h' in G such that $h'(e) = h(e)$ for all edges $e \in O_h \cup I_h$. Here O_h is the set of boundary edges e where $h(e)$ is an integer, and I_h is the set of edges contained in little triangles Δ such that h takes integer values on the three edges of Δ .*

Remark 1. One could attempt to further strengthen Theorem 1.1 by asking: can one improve any concave cocirculation h to an integer concave cocirculation preserving the values on *all* edges where h is integral? In general, the answer is negative; a counterexample will be given in the end of this paper.

Our method of proof of Theorem 1.2 is constructive and based on iteratively transforming the current concave cocirculation until the desired integer concave cocirculation is found. As a consequence, we obtain a polynomial-time combinatorial algorithm to improve h to h' as required. (The idea of proof of Theorem 1.1 in [1] is to show the existence of a concave cocirculation coinciding with h_0 on $B(G)$ whose values are expressed as integer combinations of values of h_0 ; [4] establishes an analogous property for honeycombs. Our approach is different.) We prefer to describe an iteration by considering a corresponding task on the related honeycomb model and then translating the output to the language of cocirculations in G , as this makes our description technically simpler and more enlightening.

The above theorems admit a re-formulation in polyhedral terms. Given a subset $F \subseteq E(G)$ and a function $h_0 : F \rightarrow \mathbb{R}$, let $\mathcal{C}(G, h_0)$ denote the set of concave cocirculations in G such that $h(e) = h_0(e)$ for all $e \in F$. Since concave cocirculations are described by linear constraints, $\mathcal{C}(G, h_0)$ forms a (possibly empty) polyhedron in $\mathbb{R}^{E(G)}$. Then Theorem 1.1 says that such a polyhedron (if nonempty) has an integer point h in the case h_0 is an integer-valued function on $B(G)$, whereas Theorem 1.2 is equivalent to saying that a similar property takes place if $h_0 : F \rightarrow \mathbb{Z}$ and $F = B' \cup \bigcup (E(\Delta) : \Delta \in T)$, where $B' \subseteq B(G)$ and T is a set of little triangles. (Note that when $F = B(G)$ and when $\mathcal{R}(G)$ is not a hexagon, one can conclude from the concavity that $\mathcal{C}(G, h_0)$ is bounded, i.e., it is a polytope.)

On the “negative” side, it turned out that $\mathcal{C}(G, h_0)$ with $h_0 : B(G) \rightarrow \mathbb{Z}$ need not be an integer polytope; an example with a half-integer but not integer vertex is given in [4] (and in [1]). One can show that the class of such polyhedra has “unbounded fractionality”. Moreover, denominators of vertex entries can be arbitrarily increasing as the size of G grows even if functions h_0 with smaller domains are considered. Hereinafter by the *size* of G we mean its maximum side length (=number of edges). We show the following.

Theorem 1.3 *For any positive integer k , there exists a 3-side grid G of size $O(k)$ and a function $h_0 : F \rightarrow \mathbb{Z}$, where F is the set of edges of two sides of G , such that the polyhedron $\mathcal{C}(G, h_0)$ has a vertex h satisfying: (a) $h(e)$ has denominator k for some edge $e \in E(G)$, and (b) h takes integer values on all boundary edges.*

(One can see that in this case $\mathcal{C}(G, h_0)$ is also a polytope. Note also that if h'_0 is the restriction of h to a set $F' \supset F$, then h is, obviously, a vertex of the polytope $\mathcal{C}(G, h'_0)$ as well.)

This paper is organized as follows. In Section 2 we explain the notion of honeycomb and a relationship between honeycombs and concave cocirculations. Sections 3 and 4 consider special paths (open and closed) in a honeycomb and describe a certain transformation of the honeycomb in a neighbourhood of such a path. Section 5 explains how to “improve” the honeycomb by use of such transformations and eventually proves Theorem 1.2. A construction of G, h_0 proving Theorem 1.3 is given in Section 6; it relies on an approach involving honeycombs as well. We also explain there (in Remark 3) that the set F in this theorem can be reduced further. The concluding Section 7 discusses algorithmic aspects, suggests a slight strengthening of Theorem 1.2, gives a counterexample mentioned in Remark 1, and raises an open question.

2 Honeycombs

For technical needs of this paper, our definition of honeycombs will be somewhat different from, though equivalent to, that given in [4]. It is based on a notion of pre-honeycombs, and before introducing the latter, we clarify some terminology and notation user later on. Let ξ_1, ξ_2, ξ_3 be the generating vectors as above (note that they follow anticlockwise around the origin).

The term *line* is applied to (*fully*) *infinite*, *semiinfinite*, and *finite* lines, i.e., to sets of the form $a + \mathbb{R}b$, $a + \mathbb{R}_+b$, and $\{a + \lambda b : 0 \leq \lambda \leq 1\}$, respectively, where $a \in \mathbb{R}^2$ and $b \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$. For a vector (point) $v \in \mathbb{R}^2$ and $i = 1, 2, 3$, we denote by $\Xi_i(v)$ the infinite line containing v and perpendicular to ξ_i , i.e., the set of points u with $(u - v) \cdot \xi_i = 0$. (Hereinafter $x \cdot y$ denotes the inner product of vectors x, y .) The line $\Xi_i(v)$ is the union of two semiinfinite lines $\Xi_i^+(v)$ and $\Xi_i^-(v)$ with the end v , where the rays $\Xi_i^+(v) - v$, $\mathbb{R}_+\xi_i$ and $\Xi_i^-(v) - v$ follow in the *anticlockwise* order around the origin. Any line perpendicular to ξ_i is called a Ξ_i -*line*.

By a Ξ -*system* we mean a finite set \mathcal{L} of Ξ_i -lines ($i \in \{1, 2, 3\}$) along with an *integer* weighting w on them. For a point $v \in \mathbb{R}^2$, a “sign” $s \in \{+, -\}$, and $i = 1, 2, 3$, define $w_i^s(v)$ to be the sum of weights $w(L)$ of the lines $L \in \mathcal{L}$ whose intersection with $\Xi_i^s(v)$ contains v and is a line (not a point). We call a Ξ -system (\mathcal{L}, w) a *pre-honeycomb* if for any point v , the numbers $w_i^s(v)$ are *nonnegative* and satisfy the condition

$$(2.1) \quad w_1^+(v) - w_1^-(v) = w_2^+(v) - w_2^-(v) = w_3^+(v) - w_3^-(v) =: \text{div}_w(v);$$

$\text{div}_w(v)$ is called the *divergency* at v .

Now a *honeycomb* is a certain non-standard edge-weighted planar graph $\mathcal{H} = (\mathcal{V}, \mathcal{E}, w)$ with vertex set $\mathcal{V} \neq \emptyset$ and edge set \mathcal{E} in which non-finite edges are allowed. More precisely: (i) each vertex is incident with at least 3 edges; (ii) each edge is a line with no interior point contained in

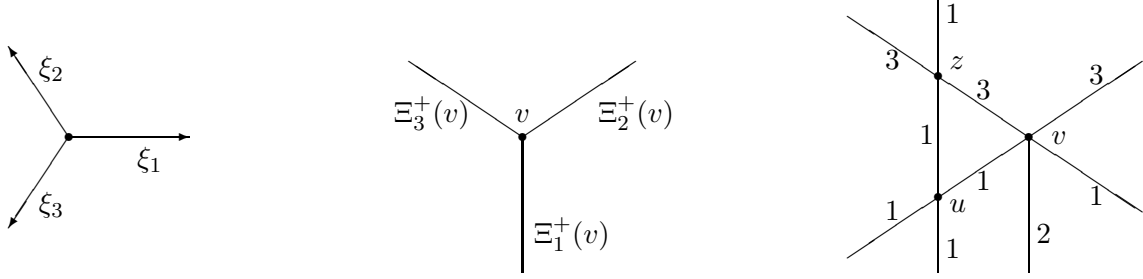


Figure 1: Generating vectors, lines $\Xi_i^+(v)$, and a honeycomb instance

another edge; (iii) $w(e)$ is a *positive* integer for each $e \in \mathcal{E}$; and (iv) (\mathcal{E}, w) is a pre-honeycomb. Then each vertex v has degree at most 6, and for $i = 1, 2, 3$ and $s \in \{+, -\}$, we denote by $e_i^s(v)$ the edge incident to v and contained in $\Xi_i^s(v)$ when such an edge exists, and say that this edge *has sign s at v* . In [4] the condition on a vertex v of a honeycomb similar to (2.1) is called the *zero-tension condition*, motivated by the observation that if each edge incident to v pulls on v with a tension equal to its weight, then the total force applied to v is zero. Figure 1 illustrates a honeycomb with three vertices u, v, z and ten edges of which seven are semiinfinite.

We will take advantage of the fact that any pre-honeycomb (\mathcal{L}, w) determines, in a natural way, a unique honeycomb $\mathcal{H} = (\mathcal{V}, \mathcal{E}, w')$ with $w_i^s(v) = (w')_i^s(v)$ for all v, s, i . Here \mathcal{V} is the set of points v for which at least three numbers among $w_i^s(v)$'s are nonzero. The set \mathcal{E} consists of all maximal Ξ_i -lines e for $i = 1, 2, 3$ such that any interior point v on e satisfies $w_i^+(v) > 0$ and does not belong to \mathcal{V} ; the weight $w'(e)$ is defined to be just this number $w_i^+(v)$ ($=w_i^-(v)$), which does not depend on v .

Since $\mathcal{V} \neq \emptyset$, one can conclude from (2.1) that a honeycomb has no fully infinite edge but the set of semiinfinite edges in it is nonempty. This set, called the *boundary* of \mathcal{H} and denoted by $\mathcal{B}(\mathcal{H})$, is naturally partitioned into subsets \mathcal{B}_i^s consisting of the semiinfinite edges of the form $\Xi_i^s(\cdot)$. Then (2.1) implies that $w(\mathcal{B}_i^+) - w(\mathcal{B}_i^-)$ is the same for $i = 1, 2, 3$. (For a subset $E' \subseteq E$ and a function $c : E \rightarrow \mathbb{R}$, $c(E')$ stands for $\sum(c(e) : e \in E')$.)

Let us introduce the *dual coordinates* d_1, d_2, d_3 of a point $x \in \mathbb{R}^2$ by setting

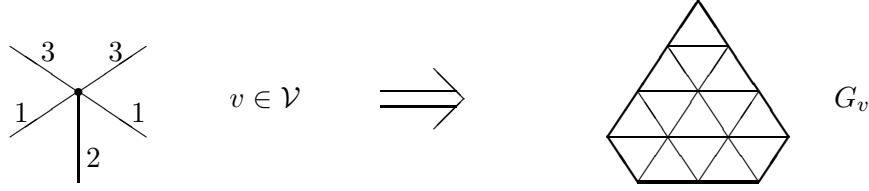
$$d_i(x) := -x \cdot \xi_i, \quad i = 1, 2, 3.$$

Since $\xi_1 + \xi_2 + \xi_3 = 0$, one has

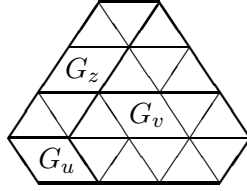
$$(2.2) \quad d_1(x) + d_2(x) + d_3(x) = 0 \quad \text{for each } x \in \mathbb{R}^2.$$

When one traverses an edge e of a honeycomb, one dual coordinate remains constant while the other two trade off; this *constant dual coordinate* is denoted by $d^c(e)$.

Next we explain that the honeycombs one-to-one correspond to the concave cocirculations via a sort of planar duality. Consider a honeycomb $\mathcal{H} = (\mathcal{V}, \mathcal{E}, w)$. Let $v \in \mathcal{V}$. Since the numbers $w_i^s(v)$ are nonnegative, condition (2.1) is equivalent to the existence of a (unique) grid G_v whose boundary is formed, in the anticlockwise order, by $w_1^+(v)$ edges parallel to ξ_1 , followed by $w_3^-(v)$ edges parallel to ξ_3 , followed by $w_2^+(v)$ edges parallel to ξ_2 , and so on, as illustrated in the picture.



For an edge e incident to v , label e^* the corresponding side of G_v . For each finite edge $e = uv$ of \mathcal{H} , glue together the grids G_u and G_v by identifying the sides labelled e^* in both. One can see that the resulting graph G is a convex grid. Also each nonempty set \mathcal{B}_i^s one-to-one corresponds to a side of G , denoted by B_i^s ; it is formed by $w(\mathcal{B}_i^s)$ edges parallel to ξ_i , and the outward normal at B_i^s points at the direction (i, s) . The picture below illustrates the grid generated by the honeycomb in Fig. 1.



The dual coordinates of vertices of \mathcal{H} generate the function h on $E(G)$ defined by:

$$(2.3) \quad h(e) := d_i(v), \quad \text{where } v \in \mathcal{V}, i = 1, 2, 3, \text{ and } e \text{ is an edge in } G_v \text{ parallel to } \xi_i.$$

Then (2.2) implies $h(e) + h(e') + h(e'') = 0$ for any little triangle with edges e, e', e'' in G , i.e., h is a cocirculation. To see that h is concave, it suffices to check (1.1) for a little rhombus ρ formed by little triangles lying in different graphs G_u and G_v . Then $\tilde{e} = uv$ is an edge of \mathcal{H} ; let \tilde{e} be perpendicular to ξ_i and assume $\tilde{e} = e_i^-(u) = e_i^+(v)$. Observe that $d_{i+1}(u) > d_{i+1}(v)$ (taking indices modulo 3) and that the side-edge e of ρ lying in G_u and parallel to ξ_{i+1} enters an obtuse vertex of ρ . Therefore, $h(e) = d_{i+1}(u) > d_{i+1}(v) = h(e')$, where e' is the side-edge of ρ parallel to e (lying in $G(v)$), as required.

Conversely, let h be a concave cocirculation in a convex grid G . Subdivide G into maximal subgraphs G_1, \dots, G_k , each being the union of little triangles where, for each $i = 1, 2, 3$, all edges parallel to ξ_i have the same value of h . The concavity of h implies that each G_j is again a convex grid; it spans a maximal region where the corresponding function g on \mathcal{R} is affinely linear, called a *flatspace* of h . For $j = 1, \dots, k$, take the point v_j in the plane defined by the dual coordinates $d_i(v_j) = h(e_i)$, $i = 1, 2, 3$, where e_i is an edge of G_j parallel to ξ_i . (The property $h(e_1) + h(e_2) + h(e_3) = 0$ implies (2.2), so v_j exists; also the points v_j are different). For each pair of graphs $G_j, G_{j'}$ having a common side S , connect v_j and $v_{j'}$ by line (finite edge) ℓ ; observe that ℓ is perpendicular to S . And if a graph G_j has a side S contained in the boundary of G , assign the semiinfinite edge $\ell = \Xi_i^s(v_j)$ whose direction (i, s) corresponds to the outward normal at S . In all cases the weight of ℓ is assigned to be the number of edges in S . One can check (by reversing the argument above) that the obtained sets of points and weighted lines constitute a honeycomb \mathcal{H} , and that the above construction for \mathcal{H} returns G, h .

3 Legal Paths and Cycles

In this section we consider certain paths (possibly cycles) in a honeycomb $\mathcal{H} = (\mathcal{V}, \mathcal{E}, w)$. A transformation of \mathcal{H} with respect to such a path, described in the next section, “improves” the honeycomb, in a certain sense, and we will show that a number of such improvements results in a honeycomb determining an integer concave cocirculation as required in Theorem 1.2. First of all we need some definitions and notation.

Let \mathcal{V}^\bullet denote the set of vertices $v \in \mathcal{V}$ having at least one nonintegral dual coordinate $d_i(v)$, and \mathcal{E}^\bullet the set of edges $e \in \mathcal{E}$ whose constant coordinate $d^c(e)$ is nonintegral. For brevity we call such vertices and edges *nonintegral*.

For $s \in \{+, -\}$, $-s$ denotes the sign opposite to s . Let $v \in \mathcal{V}$. An edge $e_i^s(v)$ is called *dominating* at v if $w_i^s(v) > w_i^{-s}(v)$. By (2.1), v has either none or three dominating edges, each pair forming an angle of 120° . A pair $\{e_i^s(v), e_j^{s'}(v)\}$ of distinct *nonintegral* edges is called *legal* if either they are *opposite* to each other at v , i.e., $j = i$ and $s' = -s$, or both edges are dominating at v (then $j \neq i$ and $s' = s$). By a *path* in \mathcal{H} we mean a finite alternating sequence $P = (v_0, q_1, v_1, \dots, q_k, v_k)$, $k \geq 1$, of vertices and edges where: for $i = 2, \dots, k-1$, q_i is a finite edge and v_{i-1}, v_i are its ends; q_1 is either a finite edge with the ends v_0, v_1 , or a semiinfinite edge with the end v_1 ; similarly, if $k > 1$ then q_k is either a finite edge with the ends v_{k-1}, v_k , or a semiinfinite edge with the end v_{k-1} . When q_1 is semiinfinite, v_0 is thought of as a *dummy* (“infinite”) vertex, and we write $v_0 = \{\emptyset\}$; similarly, $v_k = \{\emptyset\}$ when q_k is semiinfinite and $k > 1$. Self-intersecting paths are admitted. We call P

- (i) an *open* path if $k > 1$ and both edges q_1, q_k are semiinfinite;
- (ii) a *legal path* if each pair of consecutive edges q_i, q_{i+1} is legal;
- (iii) a *legal cycle* if it is a legal path, $v_0 = v_k \neq \{\emptyset\}$ and $\{q_k, q_1\}$ is a legal pair.

A legal cycle P is usually considered up to shifting cyclically and the indices are taken modulo k . We say that a legal path P *turns* at $v \in \mathcal{V}$ if, for some i with $v_i = v$, the (existing) edges q_i, q_{i+1} are not opposite at v_i (then q_i, q_{i+1} are different dominating edges at v_i). We also call such a triple (q_i, v_i, q_{i+1}) a *bend* of P at v .

Assume \mathcal{V}^\bullet is nonempty. (When $\mathcal{V}^\bullet = \emptyset$, the concave cocirculation in G determined by \mathcal{H} is already integral.) A trivial but important observation from (2.2) is that if a vertex v has a nonintegral dual coordinate, then it has at least two nonintegral dual coordinates. This implies $\mathcal{E}^\bullet \neq \emptyset$. Moreover, if e is a nonintegral edge dominating at v , then e forms a legal pair with another nonintegral edge dominating at v .

Our method of proof of Theorem 1.2 will rely on the existence of a legal path with some additional properties, as follows.

Lemma 3.1 *There exists an open legal path or a legal cycle $P = (v_0, q_1, v_1, \dots, q_k, v_k)$ such that:*

- (i) *each edge e of \mathcal{H} occurs in P at most twice, and if it occurs exactly twice, then P traverses e in both directions, i.e., $e = q_i = q_j$ and $i < j$ imply $v_i = v_{j-1}$;*
- (ii) *if an edge e occurs in P twice, then $w(e) > 1$;*
- (iii) *for each vertex v of \mathcal{H} , the number of times P turns at v does not exceed $\min\{2, |\text{div}_w(v)|\}$.*

Proof. We grow a legal path P , step by step, by the following process. Initially, choose a nonintegral semiinfinite edge e if it exists, and set $P = (v_0, q_1, v_1)$, where $v_0 := \{\emptyset\}$, $q_1 := e$, and v_1 is the end of e . Otherwise we start with $P = (v_0, q_1, v_1)$, where q_1 is an arbitrary nonintegral finite edge and v_0, v_1 are its ends. Let $P = (v_0, q_1, v_1 \dots, q_i, v_i)$ be a current legal path with $v := v_i \in \mathcal{V}$ satisfying (i),(ii),(iii). At an iteration, we wish either to increase P by adding some q_{i+1}, v_{i+1} (maintaining (i),(ii),(iii)) or to form the desired cycle.

By the above observation, $e := q_i$ forms a legal pair with at least one edge e' incident to v . We select such an e' by rules specified later and act as follows. Suppose e' occurs in P and is traversed from v , i.e., $v = v_{j-1}$ and $e' = e_j$ for some $j < i$. Then the part of P from v_{j-1} to v_i forms a legal cycle; we finish the process and output this cycle. Clearly it satisfies (i) and (ii) (but the number of bends at v may increase). Now suppose e' is not traversed from v . Then we grow P by adding e' as the new last edge q_{i+1} and adding v_{i+1} to be the end of e' different from v if e' is finite, and to be $\{\emptyset\}$ if e' is semiinfinite. In the latter case, the new P is an open legal path (taking into account the choice of the first edge q_1); we finish and output this P . And in the former case, we continue the process with the new current P . Clearly property (i) is maintained.

We have to show that e' can be chosen so as to maintain the remaining properties (concerning e' and v). Consider two cases.

Case 1. e is not dominating at v . Then e' is opposite to e at v (as the choice is unique), and (iii) remains valid as no new bend at v arises. If e' is dominating at v , then $w(e') > w(e) \geq 1$, implying (ii). And if e' is not dominating at v , then $w(e') = w(e)$ and, obviously, the new P traverses e' as many times as it traverses e , implying (ii) as well.

Case 2. e is dominating at v . Let the old P turn b times at v . First suppose P has a bend $\beta = (q_j, v_j, q_{j+1})$ at v not using the edge e . Since the edges occurring in any bend are nonintegral and dominating at the corresponding vertex, $\{e, q_{j+1}\}$ is a legal pair. We choose e' to be q_{j+1} . This leads to forming a cycle with b bends at v as before (as the bend β is destroyed while the only bend (e, v, e') is added), implying (iii).

So assume β as above does not exist. Then the old P can have at most one bend at v , namely, one of the form $\beta' = (q, v, e)$, whence $b \leq 1$. If $b < |\text{div}_w(v)|$, then taking as e' a nonintegral edge dominating at v and different from e maintains both (ii) and (iii) (to see (ii), observe that the number of times P traverses e' is less than $w(e')$). Now let $b = |\text{div}_w(v)|$. Then $b = 1$ (as $\text{div}_w(v) \neq 0$) and P has the bend β' as above. Therefore, P traverses e twice (in β' and as q_i), and we conclude from this fact together with $|\text{div}_w(v)| = 1$ that e has the opposite edge \bar{e} at v . Moreover, \bar{e} cannot occur in P . For otherwise P would traverse e more than twice, taking into account that \bar{e} forms a legal pair only with e (as \bar{e} is non-dominating at v). Thus, the choice of e' to be \bar{e} maintains (ii) and (iii), completing the proof of the lemma. ■

4 ϵ -Deformation

Let $P = (v_0, q_1, v_1, \dots, q_k, v_k)$ be as in Lemma 3.1. Our transformation of the honeycomb $\mathcal{H} = (\mathcal{V}, \mathcal{E}, w)$ in question is, roughly speaking, a result of “moving a unit weight copy of P in a normal

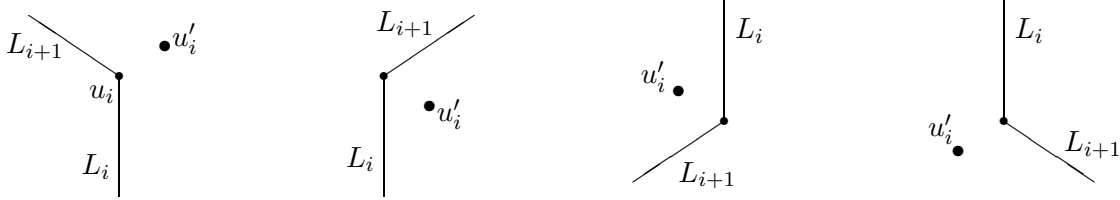
direction” (considering P as a curve in the plane); this is analogous to an operation on more elementary paths or cycles of honeycombs in [4, 5]. It is technically easier for us to describe such a transformation by handling a pre-honeycomb behind \mathcal{H} in which the line set include the maximal straight subpaths of P .

When (q_i, v_i, q_{i+1}) is a bend, we say that v_i is a *bend vertex* of P . We assume that v_0 is a bend vertex if P is a cycle. For a bend vertex v_i , we will distinguish between the cases when P *turns right* and *turns left* at v_i , defined in a natural way regarding the orientation of P . Let $v_{t(0)}, v_{t(1)}, \dots, v_{t(r)}$ ($0 = t(0) < t(1) < \dots < t(r) = k$) be the sequence of bend or dummy vertices of P . Then for $i = 1, \dots, r$, the union of edges $q_{t(i-1)+1}, \dots, q_{t(i)}$ is a (finite, semiinfinite or even fully infinite) line, denoted by L_i . For brevity $v_{t(i)}$ is denoted by u_i .

Our transformation of \mathcal{H} depends on a real parameter $\epsilon > 0$ measuring the distance of moving P and on a direction of moving; let for definiteness we wish to move P “to the right” (moving “to the left” is symmetric). We assume that ϵ is small enough; an upper bound on ϵ will be discussed later. By the transformation, a unit weight copy of each line L_i (considered as oriented from u_{i-1} to u_i) is split off the honeycomb and moves (possibly extending or shrinking) at distance ϵ to the right, turning into a parallel line L'_i connecting u'_{i-1} and u'_i . Let us describe this construction more formally. First, for a bend vertex u_i , let the constant coordinates of the lines L_i and L_{i+1} be p -th and p' -th dual coordinates, respectively. Then the point u'_i is defined by

$$(4.1) \quad \begin{aligned} d_p(u'_i) &:= d_p(u_i) - \epsilon \text{ and } d_{p'}(u'_i) := d_{p'}(u_i) + \epsilon \text{ if both } L_i, L_{i+1} \text{ have sign } + \text{ at } v, \text{ and} \\ d_p(u'_i) &:= d_p(u_i) + \epsilon \text{ and } d_{p'}(u'_i) := d_{p'}(u_i) - \epsilon \text{ if } L_i, L_{i+1} \text{ have sign } - \text{ at } v, \end{aligned}$$

where, similar to the edges, a Ξ_i -line is said to have sign s at its end v if it is contained in $\Xi_i^s(v)$. Possible cases are illustrated in the picture.



Second, for $i = 1, \dots, r$, define L'_i to be the line connecting u'_{i-1} and u'_i (when P is an open path, u'_0 and u'_r are dummy points and the non-finite lines L'_1, L'_r are defined in a natural way). Denoting by $\ell(L)$ the euclidean length (scaled by $2/\sqrt{3}$) of a line L , one can see that if a line L_i is finite and $\epsilon \leq \ell(L_i)$, then

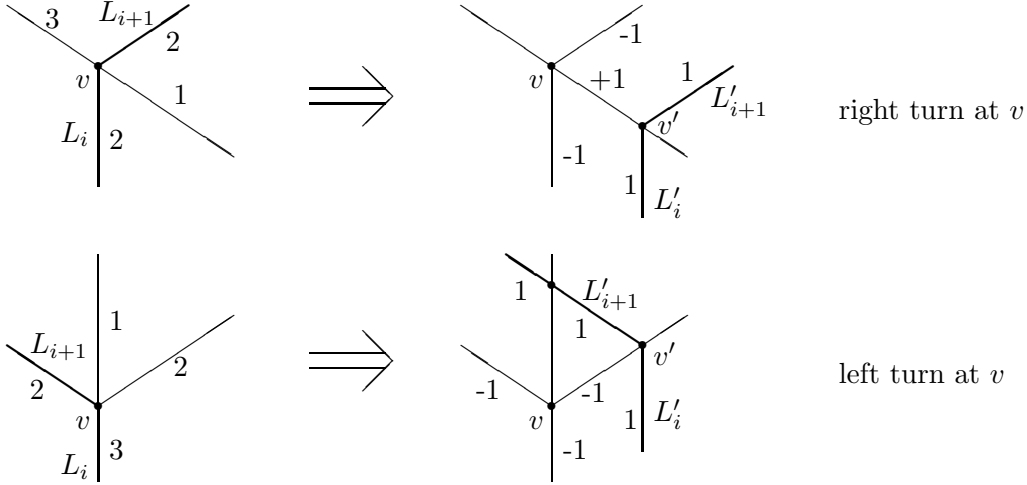
$$(4.2) \quad \ell(L'_i) = \ell(L_i) - \epsilon \text{ if } P \text{ turns right at both } u_{i-1} \text{ and } u_i, \text{ and } \ell(L'_i) \geq \ell(L_i) \text{ otherwise.}$$

This motivates a reasonable upper bound on ϵ , to be the minimum length $\bar{\epsilon}_0$ of an L_i such that P turns right at both u_{i-1}, u_i ($\bar{\epsilon}_0 = \infty$ when no such L_i exists).

Third, consider the Ξ -system $\mathcal{P} = (\{L_1, \dots, L_r\} \cup \mathcal{E}, \tilde{w})$, where $\tilde{w}(L_i) = 1$ for $i = 1, \dots, r$, and $\tilde{w}(e)$ is equal to $w(e)$ minus the number of occurrences of $e \in \mathcal{E}$ in L_1, \dots, L_r . This \mathcal{P} is a pre-honeycomb representing \mathcal{H} , i.e., satisfying $w_i^s(v) = \tilde{w}_i^s(v)$ for all v, i, s . We replace in \mathcal{P} the lines L_1, \dots, L_r by the lines L'_1, \dots, L'_r with unit weight each. (When $\epsilon = \bar{\epsilon}_0 < \infty$, the length of

at least one line L'_i reduces to zero, by (4.2), and this line vanishes in \mathcal{P}' .) Also for each bend vertex u_i , we add line R_i connecting u_i and u'_i . We assign to R_i weight 1 if P turns right at u_i , and -1 otherwise. Let $\mathcal{P}' = (\mathcal{L}', w')$ be the resulting Ξ -system.

The transformation in a neighbourhood of a bend vertex $v = u_i$ is illustrated in the picture; here the numbers on edges indicate their original weights or the changes due to the transformation, and (for simplicity) P passes v only once.



Lemma 4.1 *There exists $\bar{\epsilon}_1$, $0 < \bar{\epsilon}_1 \leq \bar{\epsilon}_0$ such that \mathcal{P}' is a pre-honeycomb for any nonnegative real $\epsilon \leq \bar{\epsilon}_1$.*

Proof. Let $0 < \epsilon < \bar{\epsilon}_0$. To see that \mathcal{P}' has zero tension everywhere, it suffices to check this property at the bend vertices u_i of P and their “copies” u'_i . For a bend vertex u_i , let \mathcal{P}_i be the Ξ -system formed by the lines L_i, L_{i+1}, R_i with weights $-1, -1, w'(R_i)$, respectively, and \mathcal{P}'_i the Ξ -system formed by the lines L'_i, L'_{i+1}, R_i with weights $1, 1, w'(R_i)$, respectively. One can see that \mathcal{P}_i has zero tension at u_i and \mathcal{P}'_i has zero tension at u'_i , wherever (right or left) P turns at u_i . This implies the zero tension property for \mathcal{P}' , taking into account that \mathcal{P} has this property (as \mathcal{P}_i and \mathcal{P}'_i describe the corresponding local changes concerning u_i, u'_i when \mathcal{P} turns into \mathcal{P}').

It remains to explain that the numbers $(w')_j^s(v)$ are nonnegative for all corresponding v, j, s when $\epsilon > 0$ is sufficiently small.

By (i),(ii) in Lemma 3.1, $\tilde{w} \geq 0$ for all $e \in \mathcal{E}$. So the only case when $(w')_p^s(v)$ might be negative is when v lies on a line R_i with weight -1 and R_i is a Ξ_p -line. Note that if $u_j = u_{j'}$ for some $j \neq j'$, i.e., P turns at the corresponding vertex v of \mathcal{H} twice, then the points u'_j and $u'_{j'}$ move along different rays out of v (this can be concluded from (4.1), taking into account (i) in Lemma 3.1). Hence we can choose $\epsilon > 0$ such that the interiors of the lines R_i are pairwise disjoint.

Consider R_i with $w'(R_i) = -1$, and let e be the edge dominating at the vertex $v = u_i$ and different from $q_{t(i)}$ and $q_{t(i)+1}$. Observe that the point u'_i moves just along e , so the line R_i is entirely contained in e when ϵ does not exceed the length of e . We show that $\tilde{w}(e) > 0$, whence the result follows. This is equivalent to saying that the number α of lines among L_1, \dots, L_r that

contain e (equal to 0,1 or 2) is strictly less than $w(e)$. For a contradiction, suppose $\alpha = w(e)$ (as $\alpha \leq w(e)$, by Lemma 3.1). This implies that the number of bends of P at v using the edge e (equal to the number of lines L_j that contain e and have one end at v) is at least $|\text{div}_w(v)|$. Then the total number of bends at v is greater than $|\text{div}_w(v)|$ (as the bend $(q_{t(i)}, u_i, q_{t(i)+1})$ does not use e), contradicting (iii) in Lemma 3.1. So $\alpha < w(e)$, as required. ■

For ϵ as in the lemma, \mathcal{P}' determines a honeycomb \mathcal{H}' , as explained in Section 2. We say that \mathcal{H}' is obtained from \mathcal{H} by the *right ϵ -deformation* of P .

5 Proof of the Theorem

In what follows, given a honeycomb with edge weights w'' , we say that vertices u, v have *the same sign* if $\text{div}_{w''}(u)\text{div}_{w''}(v) \geq 0$, and call $|\text{div}_{w''}(v)|$ the *excess* at v .

Consider a concave cocirculation h in a convex grid G and the honeycomb $\mathcal{H} = (\mathcal{V}, \mathcal{E}, w)$ determined by h . Define $\beta = \beta_{\mathcal{H}}$ to be the total weight of *nonintegral* semiinfinite edges of \mathcal{H} , $\delta = \delta_{\mathcal{H}}$ to be the total excess of *nonintegral* vertices of \mathcal{H} , and $\omega = \omega_{\mathcal{H}}$ to be the total weight of edges incident to *integral* vertices. We prove Theorem 1.2 by induction on

$$\eta := \eta_{\mathcal{H}} := \beta + \delta - \omega,$$

considering all concave cocirculations h in the given G . Observe that ω does not exceed the number of edges of G ; hence η is bounded from below. Note also that in the case $\beta = \delta = 0$, all edges of \mathcal{H} are integral (whence h is integral as well). Indeed, suppose the set \mathcal{E}^\bullet of nonintegral edges of \mathcal{H} is nonempty, and let $e \in \mathcal{E}^\bullet$. Take the maximal line L that contains e and is covered by edges of \mathcal{H} . Since $d^c(L) = d^c(e)$ is not an integer and $\beta = 0$, L contains no semiinfinite edge; so L is finite. The maximality of L implies that each end v of L is a vertex of \mathcal{H} and, moreover, $\text{div}_w(v) \neq 0$. Also $v \in \mathcal{V}^\bullet$. Then $\delta \neq 0$; a contradiction.

Thus, we may assume that $\beta + \delta > 0$ and that the theorem is valid for all concave cocirculations on G whose corresponding honeycombs \mathcal{H}' satisfy $\eta_{\mathcal{H}'} < \eta_{\mathcal{H}}$. We use notation, constructions and facts from Sections 3,4.

Choose $P = (v_0, q_1, v_1, \dots, q_k, v_k)$ as in Lemma 3.1. Note that if P is a cycle, then the fact that all bends of P are of the same degree 120° implies that there are two consecutive bend vertices u_i, u_{i+1} where the direction of turn of P is the same. We are going to apply to P the right ϵ -deformation, assuming that either P is an open path, or P is a cycle having two consecutive bend vertices where it turns right (for P can be considered up to reversing).

We gradually grow the parameter ϵ from zero, obtaining the (parameteric) honeycomb $\mathcal{H}' = (\mathcal{V}', \mathcal{E}', w')$ as described in Section 4. Let $\bar{\epsilon}_1$ be specified as the *maximum* real or $+\infty$, with $\bar{\epsilon}_1 \leq \bar{\epsilon}_0$, satisfying the assertion of Lemma 4.1. (Such an $\bar{\epsilon}_1$ exists, as if $\epsilon' > 0$ and if \mathcal{P}' is a pre-honeycomb for any $0 < \epsilon < \epsilon'$, then \mathcal{P}' is a pre-honeycomb for $\epsilon = \epsilon'$ either, by continuity and compactness.) We stop growing ϵ as soon as it reaches the bound $\bar{\epsilon}_1$ or at least one of the following events happens:

(E1) when P is an open path, the constant coordinate of some of the (semiinfinite or infinite) lines L'_1 and L'_r becomes an integer;

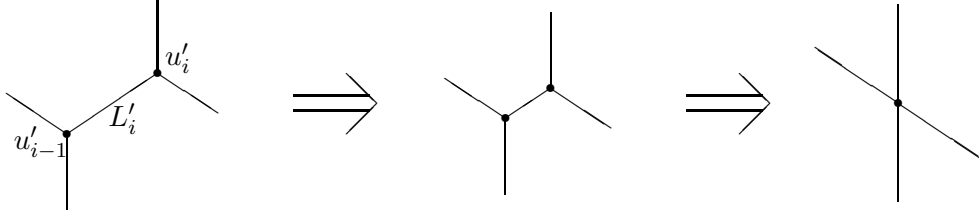
(E2) two vertices of \mathcal{H}' with different signs meet (merge);

(E3) some line L'_i meets an integer vertex v of the original \mathcal{H} .

By the above assumption, if P is a cycle, then $\bar{\epsilon}_1 \leq \bar{\epsilon}_0 < \infty$ (cf. (4.2)). And if P is an open path, then the growth of ϵ is bounded because of (E1). So we always finish with a finite ϵ ; let $\bar{\epsilon}$ and $\bar{\mathcal{H}}$ denote the resulting ϵ and honeycomb, respectively. We assert that $\eta_{\bar{\mathcal{H}}} < \eta_{\mathcal{H}}$. First of all notice that $\beta_{\bar{\mathcal{H}}} \leq \beta_{\mathcal{H}}$ (in view of (E1) and since the edges of P are nonintegral). Our further analysis relies on the following observations.

(i) When ϵ grows, each point u'_i uniformly moves along a ray from u_i , and each line L'_i uniformly moves in a normal direction to L_i . This implies that one can select a finite sequence $0 = \epsilon(0) < \epsilon(1) < \dots < \epsilon(N) = \bar{\epsilon}$ such that $N = O(|\mathcal{V}|^2)$, and for $t = 0, \dots, N-1$, the honeycomb \mathcal{H}' does not change topologically when the parameter ϵ ranges over the open interval $(\epsilon(t), \epsilon(t+1))$.

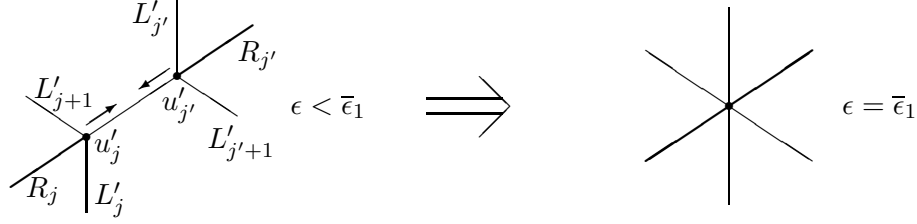
(ii) When ϵ starts growing from zero, each vertex v of \mathcal{H} occurring in P splits into several vertices whose total divergency is equal to the original divergency at v . By the construction of \mathcal{P}' and (iii) in Lemma 3.1, these vertices have the same sign. This implies that the total excess of these vertices is equal to the original excess at v . Each arising vertex u'_i has excess 1, which preserves during the process except possibly for those moments $\epsilon(t)$ when u'_i can meet another vertex of \mathcal{H}' . When two or more vertices meet, their divergencies are added up. Therefore, the sum of their excesses (before the meeting) is strictly more than the resulting excess if some of these vertices have different signs. This implies that δ reduces if (E2) happens, or if ϵ reaches $\bar{\epsilon}_0$ (since the ends of any finite line L'_i have different signs, and the vertices u'_{i-1} and u'_i meet when L'_i vanishes). The latter situation is illustrated in the picture.



(iii) Let v be an integral vertex of the initial honeycomb \mathcal{H} and let W be the total weight of its incident edges in a current honeycomb \mathcal{H}' (depending on ϵ). If, at some moment, the point v is captured by the interior of some line L'_i , this increases W by 2. Now suppose some vertex u'_i meets v . If P turns right at u_i , then W increases by at least $w'(R_i) = 1$. And if P turns left at u_i , then the lines L'_i, L'_{i+1} do not vanish (cf. (4.2)), whence W increases by $w'(L'_i) + w'(L'_{i+1}) + w'(R_i) = 1$. Therefore, ω increases when (E3) happens.

(iv) Let $\epsilon = \bar{\epsilon}_1 < \bar{\epsilon}_0$. By the maximality of $\bar{\epsilon}_1$ and reasonings in the proof of Lemma 4.1, a growth of ϵ beyond $\bar{\epsilon}_1$ would make some value $(w')^s_i(v)$ be negative. This can happen only in two cases: (a) some line R_j with weight -1 is covered by edges of \mathcal{H} when $\epsilon = \bar{\epsilon}_1$, and not covered when $\epsilon > \bar{\epsilon}_1$; or (b) some $R_j, R_{j'}$ with weight -1 each lie on the same infinite line, the points u'_j and $u'_{j'}$ move toward each other when $\epsilon < \bar{\epsilon}_1$ and these points meet when ϵ reaches $\bar{\epsilon}_1$ (then $R_j, R_{j'}$ become overlapping for $\epsilon > \bar{\epsilon}_1$). One can see that, in case (a), u'_j meets a vertex v of \mathcal{H}

(when $\epsilon = \bar{\epsilon}_1$) and the signs of v, u_j are different, and in case (b), the signs of $u_j, u_{j'}$ are different as well. In both cases, δ decreases (cf. (ii)). Case (b) is illustrated in the picture.



Using these observations, one can conclude that, during the process, β and δ are monotone nonincreasing and ω is monotone nondecreasing. Moreover, at least one of these values must change. Hence $\eta_{\mathcal{H}} < \eta_{\mathcal{H}}$, as required. (We leave it to the reader to examine details more carefully where needed.)

Let \bar{h} be the concave cocirculation in G determined by $\bar{\mathcal{H}}$. (The graph G does not change as it is determined by the list of numbers $w(\mathcal{B}_i^s)$, defined in Section 2, and this list preserves.) To justify the induction and finish the proof of the theorem, it remains to explain that

$$(5.1) \quad I_h \subseteq I_{\bar{h}} \quad \text{and} \quad O_h \subseteq O_{\bar{h}}.$$

Denote by \mathcal{H}_ϵ and h_ϵ the current honeycomb \mathcal{H}' and the induced concave cocirculation h' at a moment ϵ in the process, respectively. The correspondence between the vertices of \mathcal{H}_ϵ and the flatspaces of h_ϵ (explained in Section 2) implies that for each edge $e \in E(G)$, the function $h_\epsilon(e)$ is continuous within each interval $(\epsilon(t), \epsilon(t+1))$ (cf. (i) in the above analysis). We assert that $h' = h_\epsilon$ is continuous in the entire segment $[0, \bar{\epsilon}]$ as well.

To see the latter, consider the honeycomb $\mathcal{H}_{\epsilon(t)}$ for $0 \leq t < N$. When ϵ starts growing from $\epsilon(t)$ (i.e., $\epsilon(t) < \epsilon < \epsilon(t+1)$ and $\epsilon - \epsilon(t)$ is small) the set $Q(v)$ of vertices of $\mathcal{H}' = \mathcal{H}_\epsilon$ arising from a vertex v of $\mathcal{H}_{\epsilon(t)}$ (by splitting or moving or preserving v) is located in a small neighbourhood of v . Moreover, for two distinct vertices u, v of $\mathcal{H}_{\epsilon(t)}$, the total weight of edges of \mathcal{H}_ϵ connecting $Q(u)$ and $Q(v)$ is equal to the weight of the edge between u and v in $\mathcal{H}_{\epsilon(t)}$ (which is zero when the edge does not exist), and all these edges are parallel. This implies that for each vertex v of $\mathcal{H}_{\epsilon(t)}$, the arising subgrids $G_{v'}, v' \in Q(v)$, in G (concerning h_ϵ) give a partition of the subgrid G_v (concerning $h_{\epsilon(t)}$), i.e., the set of little triangles of G contained in these $G_{v'}$ coincides with the set of little triangles occurring in G_v . So h' is continuous within $[\epsilon(t), \epsilon(t+1))$.

Similarly, for each vertex v of $\mathcal{H}_{\epsilon(t)}$ ($0 < t \leq N$), the subgrid G_v is obtained by gluing together the subgrids $G_{v'}, v' \in Q'(v)$, where $Q'(v')$ is the set of vertices of \mathcal{H}_ϵ (with $\epsilon(t-1) < \epsilon < \epsilon(t)$) which produce v when ϵ tends to $\epsilon(t)$. So h' is continuous globally. Now since no integral vertex of the initial honeycomb can move or split during the process (but merely merge with another vertex of \mathcal{H}' if (E3) happens), we conclude that the cocirculation preserves in all little triangles where it is integral initially, yielding the first inclusion in (5.1).

The second inclusion in (5.1) is shown in a similar fashion, relying on (E1).

This completes the proof of Theorem 1.2.

6 Polyhedra $\mathcal{C}(G, h_0)$ Having Vertices with Big Denominators

In this section we describe a construction proving Theorem 1.3. We start with some definitions and auxiliary statements.

Given a concave cocirculation h in a concave grid G , the *tiling* τ_h is the subdivision of the polygon $\mathcal{R}(G)$ spanned by G into the flatspaces T of h . We also call T a *tile* in τ_h . The following elementary property of tilings will be important for us.

Lemma 6.1 *Let $F \subseteq E(G)$, $h_0 : F \rightarrow \mathbb{R}$, and $h \in \mathcal{C}(G, h_0) =: \mathcal{C}$. Then h is a vertex of \mathcal{C} if and only if h is uniquely determined by its tiling, i.e., $h' \in \mathcal{C}$ and $\tau_{h'} = \tau_h$ imply $h' = h$.*

Proof. For a little rhombus ρ of G , the fact that the sum of values of h over each of the two 3-circuits in ρ is zero implies that if rhombus inequality (1.1) turns into equality for one pair of parallel edges in ρ , then it does so for the other pair. Since \mathcal{C} is described by a system of linear constraints where the inequalities are exactly of the form (1.1), h is a vertex of \mathcal{C} if and only if it is uniquely determined by the set Q of little rhombi for which (1.1) turns into equality. Now observe that each tile in τ_h one-to-one corresponds to a component of the graph whose vertices are the little triangles of G and whose edges are the pairs of little triangles forming rhombi in Q . This implies the lemma. ■

We will use a re-formulation of this lemma involving honeycombs. Let us say that honeycombs $\mathcal{H} = (\mathcal{V}, \mathcal{E}, w)$ and $\mathcal{H}' = (\mathcal{V}', \mathcal{E}', w')$ are *conformable* if $|\mathcal{V}| = |\mathcal{V}'|$, $|\mathcal{E}| = |\mathcal{E}'|$, and there are bijections $\alpha : \mathcal{V} \rightarrow \mathcal{V}'$ and $\beta : \mathcal{E} \rightarrow \mathcal{E}'$ such that: for each vertex $v \in \mathcal{V}$ and each edge $e \in \mathcal{E}$ incident to v , the edge $\beta(e)$ is incident to $\alpha(v)$, $w(e) = w'(\beta(e))$, and if e is contained in $\Xi_i^s(v)$, then $\beta(e)$ is contained in $\Xi_i^s(\alpha(v))$. (This matches the situation when two concave cocirculations in G have the same tiling.)

Next, for $\mathcal{F} \subseteq \mathcal{E}$, we call \mathcal{H} *extreme* with respect to \mathcal{F} , or \mathcal{F} -*extreme*, if there is no honeycomb $\mathcal{H}' \neq \mathcal{H}$ such that \mathcal{H}' is conformable to \mathcal{H} and satisfies $d^c(\beta(e)) = d^c(e)$ for all $e \in \mathcal{F}$, where β is the corresponding bijection. Then the relationship between the tiling of concave cocirculations and the vertex sets of honeycombs leads to the following re-formulation of Lemma 6.1.

Corollary 6.2 *Let $F \subseteq E(G)$, $h_0 : F \rightarrow \mathbb{R}$, and $h \in \mathcal{C}(G, h_0)$. Let \mathcal{H} be the honeycomb determined by h and let \mathcal{F} be the subset of edges of \mathcal{H} corresponding to sides of tiles in τ_h that contain at least one edge from F . Then h is a vertex of $\mathcal{C}(G, h_0)$ if and only if \mathcal{H} is \mathcal{F} -extreme.*

One sufficient condition on extreme honeycombs will be used later. Let us say that a line ℓ in \mathbb{R}^2 is a *line of \mathcal{H}* if ℓ is covered by edges of \mathcal{H} . Then

- (C) \mathcal{H} is \mathcal{F} -extreme if each vertex of \mathcal{H} is contained in at least two maximal lines of \mathcal{H} , each containing an edge from \mathcal{F} .

Indeed, if two different maximal lines L, L' of \mathcal{H} intersect at a vertex v , then the constant coordinates of L, L' determine the dual coordinates of v . One can see that if \mathcal{H}' is conformable to \mathcal{H} ,

then the images of L, L' in \mathcal{H}' are maximal lines there and they intersect at the image of v . This easily implies (C).

The idea of our construction is as follows. Given a positive integer k , we will devise two honeycombs. The first honeycomb $\mathcal{H}' = (\mathcal{V}', \mathcal{E}', w')$ has the following properties:

- (P1) (i) the boundary $\mathcal{B}(\mathcal{H}')$ is partitioned into three sets $\mathcal{B}'_1, \mathcal{B}'_2, \mathcal{B}'_3$, where \mathcal{B}'_i consists of the semiinfinite edges of the form $\Xi_i^+(\cdot)$, and $w'(\mathcal{B}'_i) \leq Ck$, where C is a constant;
- (ii) the constant coordinates of all edges of \mathcal{H}' are integral;
- (iii) \mathcal{H}' is extreme with respect to $\mathcal{B}'_1 \cup \mathcal{B}'_2$.

The second honeycomb $\mathcal{H}'' = (\mathcal{V}'', \mathcal{E}'', w'')$ has the following properties:

- (P2) (i) each semiinfinite edge of \mathcal{H}'' is contained in a line of \mathcal{H}' (in particular, $d^c(e)$ is an integer for each $e \in \mathcal{B}(\mathcal{H}'')$), and $w''(\mathcal{B}(\mathcal{H}'')) \leq w'(\mathcal{B}(\mathcal{H}'))$;
- (ii) there is $e \in \mathcal{E}''$ such that the denominator of $d^c(e)$ is equal to k ;
- (iii) \mathcal{H}'' is extreme with respect to its boundary $\mathcal{B}(\mathcal{H}'')$.

Define the *sum* $\mathcal{H}' + \mathcal{H}''$ to be the honeycomb $\mathcal{H} = (\mathcal{V}, \mathcal{E}, w)$ determined by the pre-honeycomb in which the line set is the (disjoint) union of \mathcal{E}' and \mathcal{E}'' , and the weight of a line e is equal to $w'(e)$ for $e \in \mathcal{E}'$, and $w''(e)$ for $e \in \mathcal{E}''$. Then each edge of \mathcal{H} is contained in an edge of \mathcal{H}' or \mathcal{H}'' , and conversely, each edge of \mathcal{H}' or \mathcal{H}'' is contained in a line of \mathcal{H} . Using this, one can derive from (P1) and (P2) that

- (P3) (i) the boundary $\mathcal{B}(\mathcal{H})$ is partitioned into three sets $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$, where \mathcal{B}_i consists of the semiinfinite edges of the form $\Xi_i^+(\cdot)$, and $w(\mathcal{B}_i) \leq 2Ck$;
- (ii) $d^c(e) \in \mathbb{Z}$ for all $e \in \mathcal{B}(\mathcal{H})$, and $d^c(e)$ has denominator k for some edge e of \mathcal{H} ;
- (iii) \mathcal{H} is extreme with respect to $\mathcal{B}_1 \cup \mathcal{B}_2$.

(Property (iii) follows from (P1)(iii) and (P2)(i),(iii).)

Now consider the grid G and the concave cocirculation h in it determined by \mathcal{H} . By (P3)(i), G is a 3-side grid of size at most $2Ck$, with sides B_1, B_2, B_3 corresponding to $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$, respectively. Let $F := B_1 \cup B_2$ and let h_0 be the restriction of h to F (considering a side as edge set). Then (P3) together with Corollary 6.2 implies that G, h_0, h are as required in Theorem 1.3.

It remains to devise \mathcal{H}' and \mathcal{H}'' as above. To devise \mathcal{H}' is rather easy. It can be produced by truncating the dual grid formed by all lines with integer constant coordinates. More precisely, let n be a positive integer (it will depend on k). For a point $x \in \mathbb{R}^2$ and $i = 1, 2, 3$, let x^i stand for the dual coordinate $d_i(x)$. The vertex set \mathcal{V}' consists of the points v such that

$$\begin{aligned} v^i \in \mathbb{Z} \quad \text{and} \quad |v^i| < n, \quad i = 1, 2, 3, \\ v^1 - v^2 \leq n, \quad v^2 - v^3 \leq n, \quad v^3 - v^1 \leq n. \end{aligned}$$

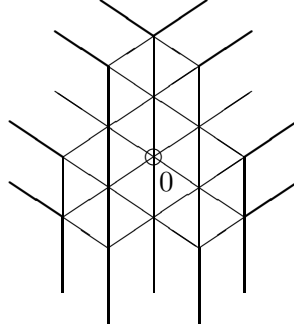
The finite edges of \mathcal{H}' have unit weights and connect the pairs $u, v \in \mathcal{V}'$ such that

$$|u^1 - v^1| + |u^2 - v^2| + |u^3 - v^3| = 2.$$

The semiinfinite edges and their weights are assigned as follows (taking indices modulo 3):

- if $v \in \mathcal{V}'$, $i \in \{1, 2, 3\}$, and $v^i - v^{i+1} \in \{n, n-1\}$, then \mathcal{H}' has edge $e = \Xi_{i-1}^+(v)$, and the weight of e is equal to 1 if $v^i - v^{i+1} = n-1$ and $v^i, v^{i+1} \neq 0$, and equal to 2 otherwise.

The case $n = 3$ is illustrated in the picture, where the semiinfinite edges with weight 2 are drawn in bold.



One can check that \mathcal{H}' is indeed a honeycomb and satisfies (P1)(i),(ii) (when $n = O(k)$). Also each vertex belongs to two lines of \mathcal{H}' , one containing a semiinfinite edge in \mathcal{B}'_1 , and the other in \mathcal{B}'_2 . Then \mathcal{H}' is $(\mathcal{B}'_1 \cup \mathcal{B}'_2)$ -expreme, by assertion (C). So \mathcal{H}' satisfies (P1)(iii) as well. Note that the set of semiinfinite edges of \mathcal{H}' is dense, in the sense that for each $i = 1, 2, 3$ and $d = -n + 1, \dots, n-1$, there is a boundary edge e of the form $\Xi_i^+(\cdot)$ such that $d^e(e) = d$.

Next we have to devise \mathcal{H}'' satisfying (P2), which is less trivial. In order to facilitate the description and technical details, we go in reverse direction: we construct a certain concave cocirculation \tilde{h} is a convex grid \tilde{G} and then transform it into the desired honeycomb.

The grid \tilde{G} spans a hexagon with S- and N-sides of length 1 and with SW-, NW-, SE-, and NE-sides of length k . We denote the vertices in the big sides (in the order as they follow in the side-path) by:

- v1. x_k, x_{k-1}, \dots, x_0 for the SW-side;
- v2. $x'_k, x'_{k-1}, \dots, x'_0$ for the NW-side;
- v3. y_0, y_1, \dots, y_k for the SE-side;
- v4. y'_0, y'_1, \dots, y'_k for the NE-side.

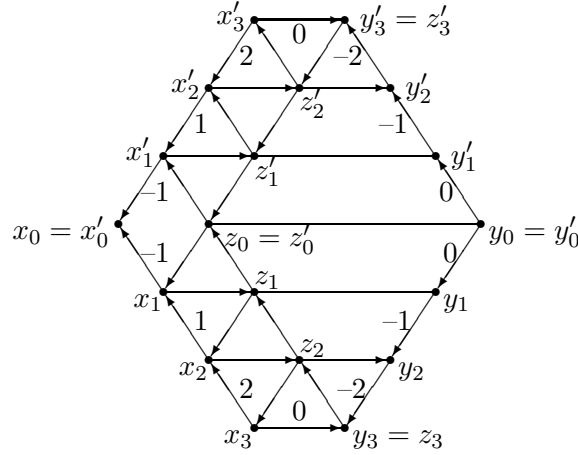
(Then $x_0 = x'_0$ and $y_0 = y'_0$.) We also distinguish the vertices $z_i := x_i + \xi_1$ and $z'_i := x'_i + \xi_1$ for $i = 1, \dots, k$ (then $z_0 = z'_0$, $z_k = y_k$, $z'_k = y'_k$.) We arrange an (abstract) tiling τ in \tilde{G} . It is symmetric w.r.t. the horizontal line x_0y_0 and consists of

- t1. $2k-2$ trapezoids and two little triangles obtained by subdividing the rhombus $R := y_k y_0 y'_k z_0$ (labeled via its vertices) by the horizontal lines passing $y_{k-1}, \dots, y_0, y'_1, \dots, y'_{k-1}$;
- t2. the little rhombus $\bar{\rho} := x_1 z_0 x'_1 x_0$;
- t3. $4k-2$ little triangles $\Delta_i := x_i z_i z_{i-1}$, $\nabla'_i := x'_i z'_{i-1} z'_i$ for $i = 1, \dots, k$, and $\nabla_j := x_j x_{j+1} z_j$, $\Delta'_j := x'_j z'_j x'_{j+1}$ for $j = 1, \dots, k-1$.

Define the function $h_0 : B(\tilde{G}) \rightarrow \mathbb{Z}$ by:

- h1. $h_0(x_i x_{i-1}) := h_0(x'_i x'_{i-1}) := i - 1$ for $i = 2, \dots, k$;
- h2. $h_0(x_1 x_0) := h_0(x'_1 x'_0) := -1$ and $h_0(x_k y_k) := h_0(x'_k y'_k) := 0$;
- h3. $h_0(y_{i-1} y_i) := h_0(y'_{i-1} y'_i) := -i + 1$ for $i = 1, \dots, k$.

The constructed τ and h_0 for $k = 3$ are illustrated in the picture (the numbers of the boundary edges indicate the values of h_0).



The tiling τ has the property that h_0 (as well as any function on $B(\tilde{G})$) is extendable to at most one cocirculation \tilde{h} in \tilde{G} such that \tilde{h} is *flat* within each tile T in τ , i.e., satisfies $\tilde{h}(e) = \tilde{h}(e')$ for any parallel edges e, e' in T . This is because we can compute \tilde{h} , step by step, using two observations: (i) the values of \tilde{h} on two nonparallel edges in a tile T determine \tilde{h} on all edges in T , and (ii) if a side S of a polygon P in \tilde{G} (spanned by a subgraph of \tilde{G}) is entirely contained in a tile, then the values of \tilde{h} on the edges of S are determined by the values on the other sides of P (since \tilde{h} is constant on S and $\tilde{h}(Q^c) = \tilde{h}(Q^a)$, where Q^c (Q^a) is the set of boundary edges of P oriented clockwise (resp. anticlockwise) around P).

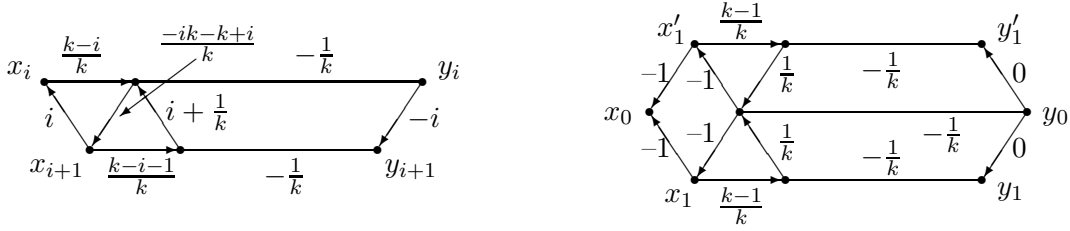
We assign \tilde{h} as follows. First assign $\tilde{h}(e) := h_0(e)$ for all $e \in B(\tilde{G})$. Second assign $\tilde{h}(z_0 x_1) := \tilde{h}(z_0 x'_1) := -1$ (by (i) for the rhombus \bar{p} in τ). Third, for an edge e in the horizontal line $z_0 y_0$, assign

$$\begin{aligned} \tilde{h}(e) &:= \frac{1}{k} \left(\tilde{h}(z_0 x_1) - \tilde{h}(x_k x_1) + \tilde{h}(x_k y_k) - \tilde{h}(y_0 y_k) \right) \\ &= \frac{1}{k} \left(-1 - (1 + \dots + (k-1)) + 0 - (0 - 1 - \dots - (k-1)) \right) = \frac{1}{k}(-1) = -1/k \end{aligned}$$

(by (ii) for the pentagon $z_0 x_1 x_k y_k y_0$, taking into account that the side $z_0 y_0$ contains k edges), where $\tilde{h}(uv)$ is the sum of values of \tilde{h} on the edges of a side uv . Therefore, $\tilde{h}(e) = -1/k$ for all horizontal edges in the big rhombus R , and we then can assign $\tilde{h}(z_{i+1} z_i) := \tilde{h}(z'_{i+1} z'_i) := i + 1/k$ for $i = 0, \dots, k-1$ (by applying (i) to the trapezoids and little triangles of τ in R). Fourth, repeatedly applying (i) to the little triangles $\Delta_1, \nabla_1, \Delta_2, \dots, \nabla_{k-1}, \Delta_k$ (in this order) that form

the trapezoid $x_1x_ky_kz_0$ in which \tilde{h} is already known on all side edges, we determine \tilde{h} on all edges in this trapezoid; and similarly for the symmetric trapezoid $z'_0y'_kx'_kx'_1$.

One can check that the obtained cocirculation \tilde{h} is well-defined, and moreover, it is a concave cocirculation with tiling τ (a careful examination of the corresponding rhombus inequalities is left to the reader). Since \tilde{h} is computed uniquely in the process, it is a vertex of $\mathcal{C}(\tilde{G}, h_0)$. Also \tilde{h} is integral on the boundary of \tilde{G} , has an entry with denominator k , and satisfies $|\tilde{h}(e)| < 2k$ for all $e \in E(\tilde{G})$. The picture indicates the values of \tilde{h} in the horizontal stripe between the lines x_iy_i and $x_{i+1}y_{i+1}$ for $1 \leq i \leq k-2$, and in the horizontal stripe between x_1y_1 and $x'_1y'_1$.



Let $\tilde{\mathcal{H}}$ be the honeycomb determined by (\tilde{G}, \tilde{h}) . It is $\mathcal{B}(\tilde{\mathcal{H}})$ -extreme, by Corollary 6.2, has all boundary edges integral and has a finite edge with constant coordinate $1/k$. We slightly modify $\tilde{\mathcal{H}}$ in order to get rid of the semiinfinite edges e of the form $\Xi_i^-(v)$ in it. This is done by truncating such an e to finite edge $e' = vu$ and adding two semiinfinite edges $a := \Xi_{i-1}^+(u)$ and $b := \Xi_{i+1}^+(u)$, all with the weight equal to that of e (which is 1 in our case). Here u is the integer point in $\Xi_i^-(v) \setminus \{v\}$ closest to v . (Strictly speaking, when applying this operation simultaneously to all such edges e , we should handle the corresponding pre-honeycomb, as some added edge may intersect another edge at an interior point.) The obtained honeycomb $\mathcal{H}'' = (\mathcal{V}'', \mathcal{E}'', w'')$ has all semiinfinite edges e of the form $\Xi_i^+(\cdot)$, with $d^c(e)$ being an integer between $-2k$ and $2k$. Also $w''(\mathcal{B}(\mathcal{H}'')) < 2|B(\tilde{G})|$. This honeycomb is extreme with respect to its boundary since so is $\tilde{\mathcal{H}}$ and since for e', a, b as above, the constant coordinate of e' is determined by the constant coordinates of a, b . Thus, \mathcal{H}'' satisfies (P2) when $n > 2k$ (to ensure that each semiinfinite edge of \mathcal{H}'' is contained in a line of \mathcal{H}').

Now (G, h) determined by the honeycomb $\mathcal{H}' + \mathcal{H}''$ is as required in Theorem 1.3.

Remark 2. In our construction of vertex \tilde{h} of $\mathcal{C}(\tilde{G}, h_0)$, the design of tiling τ is borrowed from one fragment in a construction in [2] where one shows that (in our terms) the polytope of semiconcave cocirculations in a 3-side grid with fixed integer values on the boundary can have a vertex with denominator k . Here we call a cocirculation h *semiconcave* if the inequality (1.1) holds for each rhombus ρ whose diagonal edge d is parallel to ξ_2 or ξ_3 (but may be violated if d is parallel to ξ_1).

Remark 3. The first honeycomb \mathcal{H}' in our construction turns out to be \mathcal{F}' -extreme for many proper subsets \mathcal{F}' of $\mathcal{B}(\mathcal{H}')$ and even of $\mathcal{B}'_1 \cup \mathcal{B}'_2$. For example, one can take $\mathcal{F}' := \mathcal{B}'_1 \cup \{e\}$, where e is an arbitrary edge in \mathcal{B}'_2 (a check-up is left to the reader). Moreover, it is not difficult to produce triples of boundary edges that possess such a property. For each of these sets \mathcal{F}' , the honeycomb $\mathcal{H} = \mathcal{H}' + \mathcal{H}''$ is \mathcal{F} -extreme, where \mathcal{F} consists of the semiinfinite edges of \mathcal{H} contained in members of \mathcal{F}' . Then, by Corollary 6.2, the constructed concave cocirculation h is a vertex of

the polyhedron $\mathcal{C}(G, h|_F)$ as well, where F is a subset of boundary edges of G whose images in \mathcal{H} cover \mathcal{F} (i.e., for each edge $\ell \in \mathcal{F}$ of the form $\Xi_i^+(\cdot)$, there is an $e \in F \cap B_i$ with $h(e) = d^c(\ell)$). This gives a strengthening of Theorem 1.3.

Remark 4. One can try to describe the above construction implying Theorem 1.3 so as to use the language of concave cocirculations everywhere. However, this seems to be a more intricate way. In particular, the operation on a pair h', h'' of concave cocirculations analogous to taking the sum of honeycombs $\mathcal{H}', \mathcal{H}''$ is less transparent (it is related to taking the convolution of concave functions behind h', h''). This is why we prefer to argue in terms of honeycombs.

7 Concluding Remarks

The proof of Theorem 1.2 in Section 5 provides a strongly polynomial algorithm which, given G and h , finds h' as required in this theorem. Indeed, each parameter β, δ, ω is bounded by the number of edges of G , so the number of iterations (viz. applications of the induction) is $O(n)$, where $n := |E(G)|$. As was explained, the number of moments ϵ when some line L'_i captures a vertex of \mathcal{H} or when two vertices u'_i, u'_j meet is $O(|\mathcal{V}|^2)$, or $O(n^2)$, and these moments can be computed easily. To find $\bar{\epsilon}_1$ is easy as well. Hence an iteration is performed in time polynomial in n .

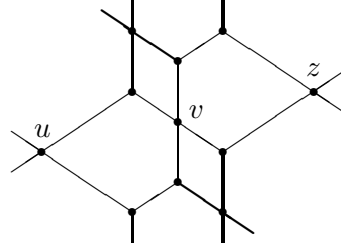
As a consequence, we obtain a strongly polynomial algorithm to solve the following problem: given a convex grid G and a function $h_0 : B(G) \rightarrow \mathbb{Z}$, decide whether h_0 is extendable to a concave cocirculation in G , and if so, find an integer concave cocirculation h with $h|_{B(G)} = h_0$. This is because the problem of finding a concave cocirculation having prescribed values on the boundary can be written as a linear program of size $O(n)$.

In view of the relationship of concave cocirculations and honeycombs, one can give an analog of Theorem 1.2 for honeycombs; we omit it here.

In fact, one can slightly modify the method of proof of Theorem 1.2 so as to obtain the following strengthening: for a concave cocirculation h in a convex grid G , there exists an integer concave cocirculation h' satisfying $h'(e) = h(e)$ for each edge $e \in O_h \cup I'_h$, where I'_h is the set of edges contained in circuits C of G such that h takes integer values on all edges of edges C . We omit the proof here.

Next, as mentioned in the Introduction, a cocirculation h in a convex grid G need not admit an improvement to an integer concave cocirculation in G preserving the values on *all* edges where h is integral. (Note that in our proof of Theorem 1.2, a vertex of the original honeycomb having only one integer dual coordinate may split into vertices not preserving this coordinate.) A counterexample (G, h) is shown in the picture (the right figure illustrates the corresponding honeycomb; here all edge weights are ones, the integral edges are drawn in bold, and each of the vertices u, v, z has one integer dual coordinate).

$$\begin{array}{ccccccc}
& & \bullet & 1 & \bullet & 0 & \bullet \\
& 0 & -1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
& \bullet & 1 & \bullet & \frac{1}{2} & \bullet & -1 \\
-\frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\bullet & 2 & \bullet & \frac{1}{2} & \bullet & 0 & \bullet \\
-\frac{1}{2} & -\frac{3}{2} & \frac{1}{2} & -1 & 1 & -1 & \\
& \bullet & 1 & \bullet & 0 & \bullet &
\end{array}$$



One can check that h is determined uniquely by its integer values, i.e., $\mathcal{C}(G, h|_F) = \{h\}$, where F is the set of edges where h is integral.

We finish with the following question motivated by some aspects in Section 6. For a convex grid G , can one give a “combinatorial characterization” for the set of tilings of concave cocirculations h such that h is a vertex of the polytope $\mathcal{C}(G, h|_{B(G)})$?

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References

- [1] A.S. Buch, The saturation conjecture (after A. Knutson and T. Tao). With an appendix by William Fulton, *Enseign. Math.* (2) **46** (2000), no. 1-2, 43–60.
- [2] J.A. De Loera and T.B. McAllister, Vertices of Gelfand-Tsetlin polytopes, *arXiv:math.CO/0309329*, 2003.
- [3] A.V. Karzanov, Concave cocirculations in a triangular grid, submitted to *Linear Algebra and Appl.*
- [4] A. Knutson and T. Tao, The honeycomb model of $GL_n(\mathbb{C})$ tensor products I: Proof of the saturation conjecture, *J. Amer. Math. Soc.* **12** (4) (1999) 1055–1090.
- [5] A. Knutson, T. Tao, and C. Woodward, The honeycomb model of $GL_n(\mathbb{C})$ tensor products II: Puzzles determine facets of the Littlewood-Richardson cone, to appear in *J. Amer. Math. Soc.*